Dear Participants and Visitors,

On behalf of the *Real Analysis Exchange*, it is an honor and a pleasure to welcome you to this, the forty-second Summer Symposium in Real Analysis. It is humbling for me to know we are here in Russia, perhaps the most important site in the development of classical real analysis, for all of our work is rooted deeply in the Russian school. This is the first year our symposium is held in Russia and I hope it won't be the last. The success of our conferences depends on a great deal of leadership and work by many people and it is wholly appropriate that I thank them and acknowledge their efforts on our behalf.

This year Yuri Andreev has provided the energy and leadership to enable this conference to be held in Russia, and I owe him a great personal debt for all his effort. Without the help of co-organizers, Sergei Kislyakov, the Director of the Steklov Institute of the Russian Academy of Sciences and Victor Budaev, the Head of Mathematics Faculty of Herzen University this conference would simply not have been possible. Too, I'm grateful to Alexander Olevskii, Valentin Skvortsov, and Maria Skopina for their essential local support, and the plenary speakers were looking forward to hearing from.

But we are here to do a little mathematics, so let this welcome end and let our work together begin.

Sincerely Yours,

Paul D. Humke
Editor-in-Chief
Real Analysis Exchange

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Contents

Plenary Talks

Vladimir Bogachov

Professor of mathematics at the Moscow State University and National Research University Higher School of Economics, Moscow, Russia

Vladimir received his Ph.D. degree from Moscow State University in 1986 under the direction of professor Smolyanov, and later, in 1991, his Dr. Sc. degree also from MSU. Since 1996 he is Professor of Math. Department at MSU. His main research interests cover much of Measure theory, Probability theory and Stochastic processes, infinite dimensional analysis. Professor Bogachev is the author of more then 170 research papers and of 11 monographs including Gaussian measures, American Math. Soc. (1998), Measure theory, 1, 2, Springer (2007), Differentiable measures and the Malliavin calculus, American Math. Soc. (2010). He has been an invited speaker at various international conferences on different branches of Analysis and Probability. He received Gold Medal from Russian Academy of Science (1990) and Award of the Japan Society for Promotion of Science.

Distributions of polynomials in many variables and Nikolskii–Besov spaces

We shall discuss recent progress in the study of spaces of fractional regularity and interesting connections of this classical area with another topic that has also been actively developing in the past decade: distributions of polynomials on spaces of high or even infinite dimension equipped with measures. One simple instance of such connections is the fact that the distribution of an arbitrary non-constant polynomial of degree d in finitely many or infinitely many Gaussian random variables belongs to the Nikolskii-Besov class of fractional order $1/d$ independently of the number of variables of the polynomial. The method of proving this rather unexpected and nontrivial fact is based on a new characterization of Besov spaces via a certain nonlinear integration by parts formula. The classical Sobolev space of functions f in L^p with first order generalized partial derivatives in L^p is equivalently described by the estimate

$$\int \varphi' f \, dx \leqslant C\|\varphi\|_q$$

for all test functions φ, where $q = p/(p-1)$. For $p = 1$ and $q = \infty$ this estimate is equivalent to the inclusion of f into the space BV of functions of bounded variation. It turns out that the Nikolskii-Besov space defined through the integral modulus of continuity

$$\int |f(x+h) - f(x)| \, dx \leqslant C|h|^\alpha$$

can be also characterized by the estimate

$$\int \varphi' f \, dx \leqslant C\|\varphi\|_\infty^\alpha \|\varphi'\|_\infty^{1-\alpha}.$$

Similar properties can be considered for other measures in place of Lebesgue measure, for example, for the Gaussian measure, which leads to important infinite-dimensional generalizations. On multidimensional spaces and manifolds one can also introduce such properties by using derivatives along vector fields in place of partial derivatives (then divergences in place of derivatives appear on the right-hand side). Yet another way of expressing such properties is connected with semigroups such as the heat semigroup and the Ornstein–Uhlenbeck semigroup. The lecture will give a concise introduction to this new direction of research including formulations of open problems. All necessary notions will be introduced and the presentation will be relatively elementary.

Natalia Kholshchevnikova

Professor of mathematics at the Moscow State Technological University "Stankin", Russia

Natalia graduated from Lomonosov Moscow State University, defended her candidate and then doctoral dissertations at the Mathematical Institute of RAS. Her research interests lie in the theory of functions, namely in the theory of uniqueness and representation of functions by series and the study of set-theoretic properties of thin sets. She was a participant of several RFBR grants, INTAS grant, a short-term Soros grant, and received a State scientific scholarship. She has participated in many international conferences.

The union problem and the category problem of sets of uniqueness in the theory of orthogonal series

The main impulse to the development of the theory of uniqueness for trigonometric series was the G.Cantor theorem (1870): *If trigonometric series converges everywhere to zero then all coefficients of this series are equal zero*; and the example of D. E. Menshov (1916),who constructed *trigonometric series that converges to zero almost everywhere but not everywhere.*

Definitions. A set $E \subset \mathbb{T}$ is called *a set of uniqueness* or U-set if every trigonometric series which converges to 0 outside E has all coefficients equal zero. Otherwise E is called *a set of multiplicity* or an M-set.

Cantor proved that empty and also finite and some countable sets are U-sets. A.Rajchman (1922) and N.K.Bary (1923) constructed perfect U-sets. Then Union and Category problems aroused.

The Union Problem *Is the union of two (countable many) "good" (Borel or analytic) sets of uniqueness also a U-set?*

The famous Theorem was proved by N.K.Bary (1923): The union of countably many closed sets of uniqueness is a set of uniqueness.

In 1981 N.N.Kholshchevnikova proved that:

1. The union of two disjoint G_δ sets of uniqueness is a set of uniqueness.

2. The union of two sets of uniqueness A and B, where B is both a G_δ and an F_σ, is a set of uniqueness.

C.Carlet and G.Debs (1985) generalised these results: The union of a sequence of U-sets E_n which are relatively closed in their union $\cup_{n=1}^{\infty} E_n$ is a U-set. The Union Problem is open for two G_δ U-sets and for G_δ U-set and a countable set. For Whalsh system analogous results were obtained by A.Shneider(1947), W.R.Wade(1971), Kholshchevnikova(1992).

The Category Problem. *Is every Borel U-set of the first category?*

This problem for trigonometric series was solved positively by G.Debs and J.Saint Raymond (1986), and for Walsh series by Kholshchevnikova (1993).

Trigonometric and Walsh systems are systems of characters on compact abelian group. The theory of uniqueness for systems of characters on zero-dimensional compact abelian group with the second axiom of countability is active developed now. V.A.Skvortcov and Kholshchevnikova (2017) solved affirmatively the category problem for such systems and obtained a generalization of the theorem on existence of a perfect M_0-set whose Hausdorff h-measure equals zero inside of any closed M_0-set.

Many interesting results were obtained in the theory of uniqueness for multiple series (trigonometric, Walsh, Haar and other) and for different type of convergence (rectangles, cubes, spheres, etc). In particular, many results that we can relate to the Union Problem were obtained by A.A.Talalyan, F.G.Arutyunyan, S.F.Lukomsky, J.M.Ash, G.V.Welland, G.Wang, Sh.T.Tetunashvili, J.Bourgain, V.A.Skvortsov, N.A.Bokaev, N.N.Kholshevnikova, M.G.Plotnikov, T.A.Sworovska, L.D.Gogoladze.

Bernd Kirchheim

Professor of mathematics at the Leipzig University, Germany

Working with David Preiss, Bernd received his PhD from the Charles University Prague in 1999 and his habilitation in 2002 while at the Max Planck Institute for Mathematics in the Sciences Leipzig. After this he had been affiliated for nearly ten years with the University of Oxford, before returning in 2012 to Leipzig. He was awarded the Whitehead prize of the LMS and is mainly interested in geometric measure theory and the (vectorial) Calculus of Variations.

Derivatives and their geometry

What is the range of the derivative of a (lipschitz) map, more precisely the possible distributions of it. This question was motivated by variational problems, where the direct method fails. The methods developed to answer this found many applications - from fluid dynamics to the study of surfaces.

We will discuss the underlying geometry in matrix space, where appropriate convexity notions give the crucial inside, and recent progress on homogeneous functionals. Some remaining geometrical question with impact on the regularity of surfaces will be presented.

Sergei Konyagin

Chief scientific researcher at the Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia

Sergei received his Ph.D. degree from Moscow State University in 1982 and his Dr. Sc. degree in 1989 also from MSU. He is a full member of Russian Academy of Sciences, a prominent specialist in Harmonic Analysis, Theory of Functions, Number Theory. Konyagin solved well known Littlewood Conjecture on estimation from below of exponential sums, long-standing Lusin problem on the representation of the function by a convergent trigonometric series, and recently he, jointly with K. Ford, B. Green, J. Maynard, and T. Tao, obtained some results in number theory related to long gaps between primes. Konyagin was awarded the Salem Prize, Vinogradov Prize of Russian Academy of Sciences. In 2012 he became a fellow of the American athematical Society.

Convergence to zero of exponential sums with positive integer coefficients and approximation by sums of shifts of a single function on the line

Petr Borodin, Sergei Konyagin

Moscow State University, Russia; Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

We prove that there is a sequence of trigonometric polynomials with positive integer coefficients, which converges to zero almost everywhere. We also prove that there is a function $f\colon \mathbb{R} \to \mathbb{R}$ such that the sums of its shifts are dense in all real spaces $L_p(\mathbb{R})$ for $2 \leqslant p < \infty$ and also in the real space $C_0(\mathbb{R})$.

Evgeny Semenov

Professor of mathematics at the Voronezh State University, Russia

Evgeny received his Ph.D. from Voronezh University (1966) under the direction of Selin Krein and later, in 1968, his Dr. Sc. diploma also from the same university. He is the author of more than 200 papers in Functional analysis, Fourier analysis, Operator theory Approximations and expansions, Convex and discrete geometry, Global analysis. He has written also some monographs and textbooks including "Interpolation of Linear Operators" (Moscow, 1978), "Geometry of Functional Spaces" (Novosibirsk, 1979), "Haar Series and Linear Operators" (Dordrecht: Kluwer, 1997). He was a speaker in many international conferences in Analysis.

On strictly singular operators

Francisco Hernandez, Evgeny Semenov

Complutense University of Madrid, Spain; Voronez State University, Russia

A linear operator A between two Banach spaces E and F is called *strictly singular (SS)* if A fails to be an isomorphism on any infinite–dimension subspace of E. This concept was introduced by T. Kato. A stronger notion was introduced by B. Mityagin and A. Pelczynski. An operator A from E to F is called *super strictly singular (SSS)* if the sequence of Bernstein widths $b_n(A)$ tends to 0 when $n \to \infty$, where

$$b_n(A) = \sup_{Q \subset E, \dim Q = n} \inf_{x \in Q, \|x\|_E = 1} \|Ax\|_F.$$

In the context of Banach lattices a weaker notion was introduced by F. L. Hernandez and B. Rodriguez-Salinas. An operator A from a Banach lattice E to a Banach space F is said to be *disjointly strictly singular (DSS)* if there is no disjoint sequence of non–null vectors (x_n) in E such that the restriction of A to the subspace $[x_n]$ is an isomorphism. It is clear that $K \subset SSS \subset SS \subset DSS$, where K denotes the set of compact operators.

We present some classical and modern results about these operator sets.

Special Session

Different levels of smoothness: Restriction, extension, and covering theorems

Chris Ciesielski

West Virginia University and University of Pennsylvania, USA

Chris received his PhD from Warsaw University, Poland in 1985 and was awarded the *Kazimierz Kuratowski Prize* from the Polish Mathematical Society and Polish Academy of Sciences in 1986. Shortly thereafter he moved to West Virginia University where he has served as Professor of Mathematics ever since. He has enjoyed a long and productive research career publishing in the areas of real analysis and its applications, foundations of mathematics including set theory, topology, logic and image processing, especially image segmentation. In 1999 he was named recipient of the *Benedum Distinguished Scholar Award*.

This sequence of two one-hour talks is based on the expository paper [1], written with Juan B. Seoane–Sepúlveda.

The aim of these talks is to revisit the centuries old discussion on the interrelations between continuous and differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, as well as their higher level analogs. The new angle of this presentation is influenced by a series of very recent results in this research area, including 2016-18 articles [2-11].

Talk 1. Differentiability versus continuity: Restriction and extension theorems and monstrous examples

This is presented in an narrative that answers two classical questions: (1) *To what extend a continuous function must be differentiable?* and (2) *How strong is the assumption of differentiability of a function?*

Question (2) will be interpreted as: *To what extent the derivative F' of an $F\colon \mathbb{R} \to \mathbb{R}$ must be continuous?* Here we recall some well known properties of the derivatives (large set of points of continuity, Darboux property) as well as newer (e.g., a finite composition of derivatives from $I = [0, 1]$ to I has fixed point property). We will also provide a very easy new construction of *everywhere differentiable nowhere monotone map.*

Concerning question (1): we indicate a simple new proof that *for every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a perfect set $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable*; discuss Jarník and Whitney differentiable extension theorems; deduce that *for every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ such that $f \cap g$ is uncountable.* We will also present a new seemingly paradoxical example a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ (which can be nowhere monotone) and of compact perfect $\mathfrak{X} \subset \mathbb{R}$ such that $F'(x) = 0$ for all $x \in \mathfrak{X}$ while $F[\mathfrak{X}] = \mathfrak{X}$; thus, the map $\mathfrak{f} = F \restriction \mathfrak{X}$ is shrinking at every point while, paradoxically, not globally.

Talk 2. Higher level differentiability: Generalized Ulam-Zahorski problem and small coverings by smooth maps

Recall that $C^\infty \subsetneq \cdots \subsetneq C^n \subsetneq D^n \subsetneq \cdots \subsetneq C^1 \subsetneq D^1 \subsetneq C^0$, where D^n is the class of all n-times differentiable $f\colon \mathbb{R} \to \mathbb{R}$ and C^n of all $f \in D^n$ whose nth derivative is continuous. For $\mathcal{F} \subsetneq \mathcal{G}$ among these classes, Ulam-Zahorski *$\mathcal{G}$-$\mathcal{F}$ interpolation* problem ask if for every $f \in \mathcal{F}$ there is a $g \in \mathcal{G}$ with $f \cap g$ uncountable. We will discuss a new result

- D^n-C^n strong interpolation theorem: *For every $f \in D^n$ and perfect $P \subset \mathbb{R}$ there is a $g \in C^n$ such that $(f \cap g) \restriction P$ is uncountable*

and show that this and earlier results solve all $\mathcal{G}$-$\mathcal{F}$ interpolation problems except for D^1-D^2 interpolation, which is open. Next, we will turn our attention to the following new result:

- Let $A \subset^\star B$ mean "$B \setminus A$ has of cardinality $\leqslant \omega_1$." *It is consistent with ZFC, follows from CPA, that for every $\nu \in \omega \cup \{\infty\}$ there exists a family $\mathcal{F}_\nu \subset C^\nu(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that*

  (i) $g \subset^\star \bigcup \mathcal{F}_\nu$ for every $g \in D^\nu(\mathbb{R})$.

  Moreover, for $n \in \{0, 1\}$, and only such n, we also have

  (ii) $g \subset^\star \bigcup \mathcal{F}_n$ for every $g \in D^n(X)$, where $X \subset \mathbb{R}$ is arbitrary.

The proof of (ii) is based on the following theorem of independent interest (which, for $\nu \neq 0$, seems to have been previously unnoticed):

- For every $X \subset \mathbb{R}$ with no isolated points, every ν-times differentiable function $g\colon X \to \mathbb{R}$ admits a ν-times differentiable extension $\bar{g}\colon B \to \mathbb{R}$, where $B \supset X$ is a Borel subset of $\mathbb{R}$.

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Contributed Talks

On the sets of fixed points, periodic functions points, recurrent points and chain recurrent points of bounded Baire one functions

Aliasghar Alikhani-Koopaei

Pennsylvania State University, USA

In this talk we give a brief history of the size of some sets related to the dynamics of continuous functions and present some results related to the sets of fixed points, periodic points, recurrent points and chain recurrent points of bounded Baire one functions.

Quasi-linear functionals on locally compact spaces

Svetlana Butler

University of California, Santa Barbara, USA

We study positive quasi-linear functionals on locally compact spaces. Then we establish a correspondence between topological measures and positive quasi-linear functionals. We show that there is an order-preserving bijection between compact-finite quasi-linear functionals and compact-finite topological measures, which is an order-preserving isometric isomorphism when quasi-linear functionals and topological measures have finite norms. We further study properties of quasi-linear functionals and give an explicit example of a quasi-linear functional.

Equilateral weights on subsets of $\mathbb{R}^n$

Emmanuel Chetcuti
University of Malta, Malta

A group-valued function f defined on a subset D of $\mathbb{R}^n$ is called an *equilateral weight* if the sum of the function values taken at the vertices of any (full-dimensional) regular simplex contained in D is the same; this constant is called the *weight* of the function. By an elementary argument, it easily follows that when $D = \mathbb{R}^n$, any equilateral weight must be constant. However, when restricting the domain D, there may exist appropriate functions which are not constant.

The aim of this presentation is to describe equilateral weights for a number of subsets of $\mathbb{R}^n$.

1. When D equals the sphere in $\mathbb{R}^n$ with radius $1/\sqrt{2}$, denoted by $S^n_{1/\sqrt{2}}$, every full-dimensional regular simplex corresponds to an orthogonal bases of $\mathbb{R}^n$ and the complete characterization of $\mathbb{R}$-valued equilateral weights on $S^n_{1/\sqrt{2}}$ is precisely the formulation of the celebrated Gleason Theorem [2].

2. Elementary but elegant combinatorial arguments show that the situation changes drastically when for D we take $S^n_{1/\sqrt{2}} \cap \mathbb{Q}^n$ [3].

3. Surprisingly, on the closed unit ball B^n of $\mathbb{R}^n$, there are no non-trivial equilateral weights [1].

4. Finally, we shall discuss $\mathbb{Z}_2$-valued weights on $S^n_{1/\sqrt{2}}$ giving an answer for $n \geqslant 4$ and state the problem left unsolved for the case when $n = 3$ [4].

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On generalized statistical and ideal convergence of metric-valued sequences

Pratulananda Das (speaker), Ekrem Savas

Jadavpur University, Kolkata, India; Istanbul Ticaret University, Istanbul, Turkey

We consider the notion of generalized density, namely, the natural density of weight g recently introduced in [1] and primarily study some sufficient and almost converse necessary conditions for the generalized statistically convergent sequence under which the subsequence is also generalized statistically convergent. Some results are also obtained in a more general form by using the notion of ideals. The entire investigation is performed in the setting of general metric spaces extending the recent results of [2]. All the results obviously contain the case of real sequences as a special case.

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On convergence of graphons

Martin Doležal

Institute of Mathematics of the Czech Academy of Sciences, Prague, Czech Republic

One possible compactification of the space of all finite graphs consists of so called graphons. We show that there are interesting connections of the compactness of graphons with the Vietoris topology of the hyperspace over functions equipped with the weak* topology.

Homeomorphisms of Hashimoto topologies

Małgorzata Filipczak (speaker), Grażyna Horbaczewska
Łódź University, Poland

We investigate homeomorphisms of different types of Hashimoto topologies based on the Euclidean topology on the real line and classic sigma-ideals.

Some dynamical and order properties of maps related to the real line

Aleksandr Florinskiy (speaker), Sergey Brygin
Saint-Petersburg State University, Russia

We consider mappings acting in the real line or some related spaces, for example, the space of all compact subsets of the line. We discuss different kinds of unboundedness of orbits of points and sets, the existence of smooth maps with prescribed orbits, relations between dynamical and order properties of functions.

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On a strong generalized topology with respect to the outer Lebesgue measure

Jacek Hejduk (speaker), Anna Loranty
Łódź University, Poland

The classical density topology is a topology generated by a lower density operator connected with a density point of a set. This topology has been studied by many mathematicians. Many generalizations of this topology, related to the generalizations of the concept of density points, are known. One of the possible generalizations of the concept of a density point is replacing the Lebesgue measure by the outer one. Unfortunately, it has turned out that the suitable family associated with such generalized density points is not a topology. However, in this case, one can prove that such a family is a strong generalized topology. A strong generalized topological space, introduced by Á. Császár, is a family $\mathcal{F}$ of subset of a nonempty set X such that empty set and X belong to the family $\mathcal{F}$ and the union of any subfamily of the family $\mathcal{F}$ belongs to $\mathcal{F}$.

During the talk some properties of a strong generalized topology connected with density points with respect to the outer Lebesgue measure will be presented. Moreover, among others, some characterizations of the families of meager sets and compact sets in such space will be given. The connection between continuous functions with respect to the above-describe topology and approximately continuous like functions connected with outer density points will be presented.

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On sets which can be moved away from sets of a certain family

Grażyna Horbaczewska (speaker), Sebastian Lindner

Łódź University, Poland

An operation which assigns to an arbitrary family of sets the class of sets which can be translated away from every set from the fixed family is considered in abelian groups. Assuming CH it is proven that on the real line meager sets can be defined as sets "shiftable" from the family of strong measure zero sets (K = SMZ*). A similar result is shown for Lebesgue null sets and strongly meager sets (N = SM*).

The Vitali convergence theorem for distribution-based nonlinear integrals

Jun Kawabe

Shinshu University, Japan

The Lebesgue integral is used for aggregating an infinite number of inputs into a single output value and is continuous with respect to inputs by the Lebesgue convergence theorem. This continuity is a guarantee for certain robustness and consistency in the aggregation process.

The Choquet, the Sugeno, and the Shilkret integrals may be considered as nonlinear aggregation functionals $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$, where $\mathcal{M}(X)$ is the set of all nonadditive measures $\mu\colon \mathcal{A} \to [0,\infty]$ on a measurable space $(X,\mathcal{A})$ and $\mathcal{F}^+(X)$ is the set of all $\mathcal{A}$-measurable functions $f\colon X \to [0,\infty]$. For those functionals, their continuity corresponds to the convergence theorem of integrals, which means that the limit of the integrals of a sequence of functions is the integral of the limit function. Thus many attempts have been made to formulate the monotone, the bounded, and the dominated convergence theorems for the Choquet, the Sugeno, and the Shilkret integrals, all of which are determined through the μ-decreasing distribution function $G_\mu(t) := \mu(\{f \geqslant t\})$.

The purpose of this talk is to present the Vitali convergence theorem for such distribution-based nonlinear integrals. A key ingredient is a perturbation of functional that manages the change in the functional value $I(\mu, f)$ when the integrand is slightly shifted from f to $f + \varepsilon$ and the μ-decreasing distribution function is slightly shifted from $G_\mu(f)$ to $G_\mu(f) + \delta$.

Multivariate sampling-type expansions

Aleksandr Krivoshein

Saint-Petersburg State University, Russia

Approximation properties of expansions $\sum_k c_k \varphi(M^j \cdot -k)$ are studied, where M is a matrix dilation, c_k are sampled values of f, i.e. $f(M^{-j}k)$, or sampled values of an appropriate differential operator L, i.e. $Lf(M^{-j}\cdot)(k)$, or the integral averages of f near $M^{-j}k$. Error estimations in L_p-norm, $p \geqslant 2$, are given in terms of the Fourier transform of f. The approximation order depends on how smooth is f, on the order of Strang-Fix condition for φ and on M. Some special properties of φ are required, but the class of functions φ we consider is large enough (including compactly supported splines as well as band-limited functions). Periodic case is also discussed.

Unconditional convergence for wavelet frame expansions

Elena Lebedeva

Saint-Petersburg State University, Russia

We study unconditional convergence for wavelet frame expansions in $L_p(\mathbb{R})$.

Let $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$, $\{\tilde{\psi}_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ be dual wavelet frames in $L_2(\mathbb{R})$, let η be an even, bounded, decreasing on $[0, \infty)$ function such that $\int\limits_0^\infty \eta(x)\ln(1+x)\,dx < \infty$, and $|\psi(x)|, |\tilde{\psi}(x)| \leqslant \eta(x)$. Than the series $\sum_{j,k\in\mathbb{Z}}(f, \tilde{\psi}_{j,k})\psi_{j,k}$ is unconditional convergent in $L_p(\mathbb{R})$, $1 < p < \infty$.

On dynamical systems, entropy and certain classes of real functions

Anna Loranty (speaker), Ryszard Pawlak

Łódź University, Poland

It is commonly accepted that if entropy of a function is positive, the function is chaotic. The analysis of different examples of functions leads us to the interesting observation that chaos, and therefore also entropy of a function, may be focused around one point. There are a lot of theories, which emphasize the importance of the problem connected with focusing entropy of a function on a set or at a point. It seems appropriate to assume that the essence of such points should be connected with a behavior of a function exclusively around this point.

During the research related to points focusing entropy it turned out that there are functions which “attracts” positive entropy at a point. This means that each function “lying near” given function has a positive entropy on every open neighborhood of a given point. During the talk issues connected with points focusing entropy and functions attracting positive entropy will be presented.

Many studies concerning dynamical systems are strictly connected with asymptotic properties of orbits. In particular, the basic questions regard the behavior of orbits of points lying close to a fixed point of a dynamical system. It leads to the issue of stability. During the lecture we will also concentrate on some kind of stability at a point.

Usually, studies related to dynamical systems are connected with systems consisting of continuous functions. During the lecture, in addition to continuous functions, almost continuous functions (in the sense of Stallings) and approximately continuous functions will also be considered.

Marginals and the product strong lifting problem

Nicolaos Macheras
University of Piraeus, Greece

The subject of the talk is related with the product strong lifting problem. Solutions to the positive are reduced to the existence of marginals with respect to product probability spaces between the ordinary product and the product whose probability measure is the restriction of the skew product of the factor probabilities to the σ-algebra obtained by adjoing either the right or left nil-null sets to the ordinary product algebra. We discuss the problem of the existence of (strong) marginals. As an application it follows, that the Radon product of a hyperstonian space associated to a Polish space with itself admits a product strong lifting.

On some lower bounds for Kolmogorov widths

Yuri Malykhin
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We will talk about lower bounds for Kolmogorov widths of some functional classes (Besov, Holder-Nikolskii, Sobolev) and corresponding finite-dimensional sets (skewed octahedra, product of octahedra, etc).

Monotonically boundedly completeness of the Franklin system in $C[0,1]$ and $L^1[0,1]$

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A basis $\{e_n\}_{n=0}^{\infty}$ of Banach space X is said to be *boundedly complete* if for every sequence $\{a_n\}_{n=0}^{\infty}$ of scalars with $\sup\limits_{n\in\mathbb{N}} \left\| \sum\limits_{k=0}^{n} a_k e_k \right\| < +\infty$, the series $\sum\limits_{n=0}^{\infty} a_n e_n$ converges. If a space possesses a boundedly complete basis, then the space is isomorphic to a dual space. In particular $C[0,1]$ and $L^1[0,1]$ do not have boundedly complete bases. Trying to find a weaker property, which may accrue for basis in nondual spaces, J.R. Holub introduced in [1] the following concept:

Definition. A semi-normalized basis $\{e_n\}_{n=0}^{\infty}$ of a Banach space X is said to be *monotonically boundedly complete* if whenever $\{a_n\}_{n=0}^{\infty}$ is a sequence of scalars which decreases monotonically to zero and for which $\sup_{n\in\mathbb{N}} \left\| \sum_{k=0}^{n} a_k e_k \right\| < +\infty$, then $\sum_{n=0}^{\infty} a_n e_n$ converges.

He proved, in particular, that the Schauder's basis in $C[0,1]$ is monotonically boundedly complete and asked whether the Haar basis in $L^1[0,1]$ and Franklin basis in $C[0,1]$ are monotonically boundedly complete as well. In [2] V. Kadets proved the monotonically boundedly completeness for Haar basis. We proved the monotonically boundedly completeness of Franklin system in $C[0,1]$ and $L^1[0,1]$ and also we prove stronger property than the monotonically boundedly completeness of Franklin system in $L^1[0,1]$.

Using some results obtained in [3] − [7], we proved the following theorems

Theorem 1. *The normalized Franklin basis for $C[0,1]$ is monotonically boundedly complete.*

Theorem 2. *The normalizes Franklin basis for $L^1[0,1]$ is monotonically boundedly complete.*

This theorem follows from theorem 3.

Theorem 3. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that*

$$\frac{|a_n|}{n^\alpha} \leqslant C_6 \frac{|a_k|}{k^\alpha}, \; n \geqslant k$$

for some $\alpha \geqslant 0$. If $\sup_{n\in\mathbb{N}} \left\| \sum_{k=0}^{n} a_k f_k \right\|_1 < +\infty$, where $\{f_n\}_{n=0}^{\infty}$ is the Franklin system normalized in $L^2[0,1]$, then $\sum_{n=0}^{\infty} a_n f_n$ converges in $L^p[0,1]$, for all $1 \leqslant p < \infty$.

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Structured population model with diffusion in structure space

Fabio Milner (speaker), Andrea Pugliese

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A structured population model is described and analyzed, in which individual dynamics may be stochastic. The model consists of a PDE of advection-diffusion type in the structure variable. The population may represent, for example, the density of infected individuals structured by pathogen density x, $x \geqslant 0$. The individuals with density $x = 0$ are not infected, but rather susceptible or recovered. Their dynamics is described by an ODE with a source term that is the exact flux from the diffusion and advection as $x \to 0^+$. Infection/reinfection is then modeled by distributing some fraction of these individuals into the infected class, but distributed in the structure variable through a probability density function. Existence of a global-in-time solution is proven, as well as a classical bifurcation result about equilibrium solutions: a net reproduction number R_0 is defined that separates the case of only the trivial equilibrium existing when $R_0 < 1$ from the existence of another —notrivial— equilibrium when $R_0 > 1$. Numerical simulation results are provided to show the stabilization towards the positive equilibrium when $R_0 > 1$ and towards the trivial one when $R_0 < 1$, result that is not proven analytically. Simulations are also provided to show the Allee effect that helps boost population sizes at low densities.

Means of iterates

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We determine continuous bijections f, acting on a real interval into itself, whose k-fold iterate is the quasi-arithmetic mean of all its subsequent iterates from f^0 up to f^n (where $0 \leqslant k \leqslant n$). Namely, we prove that if at most one of the numbers k, n is odd, then such functions consist of at most three affine pieces.

U- and V-sets for multiple Vilenkin series

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Let $\{f_n\}$ be an orthonormal system in a Hilbert space $L_2(X)$. Then a set $A \subset X$ is called a *$\mathcal{V}$-set* for the system $\{f_n\}$ if convergence of a series $\sum_n a_n f_n(x)$ to a finite summable function f on $X \setminus A$ implies that this series is the Fourier series of f. Setting $f = 0$ on $X \setminus A$, we get the definition of a *$\mathcal{U}$-set* for the system $\{f_n\}$. Each $\mathcal{V}$-set is evidently a $\mathcal{U}$-set.

We study analysis on Vilenkin groups G, i.e., on zero-dimensional second-countable compact commutative groups (see [1]). The elements of the dual group of G form an orthonormal system $\{f_n\}$ in $L_2(G)$.

Harris proved [4] that any closed, measure zero subgroup of a Vilenkin group is a $\mathcal{U}$-set. Grubb found another examples of $\mathcal{U}$-sets and de-facto proved that any closed $\mathcal{U}$-set is a $\mathcal{V}$-set (see, for example, [2, 3]). In [5] some category properties of $\mathcal{U}$-sets are established.

In the multidimensional case, examples of countable $\mathcal{U}$-sets for square convergence are constructed in [6]. We introduce a multidimensional analog of Dirichlet sets in the product of Vilenkin groups and prove that all translations of such sets are $\mathcal{V}$-sets and therefore $\mathcal{U}$-sets. The full sequence of rectangular partial sums and restricted rectangular convergence are considered. The main tool of our investigation is quasi-measures and the concept of Γ-continuity of ones.

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On rearranged multiple Haar series

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Almost 150 years ago Cantor proved that *if a trigonometric series* $TS = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ *everywhere converges to zero then* TS *is the trivial series.* We say that *uniqueness holds* for a system of functions and for some summation method if the analog of the Cantor theorem is true.

Consider the multivariate Haar system $\{\prod_{i=1}^{d} H_{n_i}(x_i)\}$. Skvortsov proved [1] that uniqueness holds for rectangular convergence. In contrast, Plotnikov established that uniqueness does not hold for square convergence. Moreover, for λ-convergent (= convergent over λ-regular rectangles) double Haar series, uniqueness does not hold if $\lambda < \sqrt{2}/2$ but holds if $\lambda > \sqrt{2}/2$ (see [2]).

For some rearrangements of the system $\{H_n(x) H_m(y)\}$, non-trivial everywhere λ-convergent series exist even if $\lambda < 2$ (see [3]). The constant 2 is the best if we consider some natural class of rearrangements. The following is true.

Theorem 1. *Let* T *be any permutation of the set* $\{0, 1, \ldots\}$ *such that* T *preserves all dyadic blocks* $\{2^{k-1}, \ldots, 2^k - 1\}$. *Then uniqueness holds for the system* $\left\{H_{T(n)}(x)\, H_{T(m)}(y)\right\}_{n,m=0}^{\infty}$ *under* λ*-convergence whenever* $\lambda \geqslant 2$.

In conclusion, we discuss some uniqueness problems concerning multiple series with respect to other systems of functions. In the case of rectangle convergence, uniqueness holds for the trigonometric system [4], for the Walsh system [1], and for the Franklin system [5]. Uniqueness also holds for spherical convergent multiple trigonometric series [6]. But the cases of square convergence and λ-convergence lead to open questions, even in dimension 2.

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On some trigonometric polynomials with extremally small uniform norm

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For $f \in L^p(0, 2\pi)$ we set

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{[0,2\pi]} |f(x)| \qquad \text{for } p = \infty.$$

Let n be a natural number. Denote by E_n the space of real trigonometric polynomials of the form

$$t(x) = \sum_{k=2^{n-1}}^{2^n - 1} a_k \cos kx + b_k \sin kx.$$

We prove the following result (see [1]).

Theorem. *Let $\varepsilon \in (0, 1)$ be a real number, let $1 \leqslant k_1 < k_2 < \ldots < k_n < \ldots$ be a sequence of natural numbers, let L_n be a subspace of E_{k_n} such that $\dim L_n \geqslant \varepsilon \dim E_{k_n}$, $n = 1, 2, \ldots$. Then there exist trigonometric polynomials $t_n \in L_n$, $n = 1, 2, \ldots$, such that $\|t_n\|_\infty \leqslant 1$, $\|t_n\|_1 \geqslant c(\varepsilon)$, $n = 1, 2, \ldots$, and*

$$\Big\| \sum_{j=1}^{n} t_j \Big\|_\infty \leqslant \sqrt{n}, \quad n = 1, 2, \ldots, \tag{1}$$

where $c(\varepsilon) > 0$ is a constant which depends only on ε.

Note that if trigonometric polynomials $\{t_n(x)\}_{n=1}^{\infty}$ are orthogonal and $\|t_n\|_1 \geqslant \alpha > 0$, $n = 1, 2, \ldots$, then

$$\Big\| \sum_{j=1}^{n} t_j \Big\|_\infty \geqslant \frac{1}{\sqrt{2\pi}} \Big\| \sum_{j=1}^{n} t_j \Big\|_2 = \frac{1}{\sqrt{2\pi}} \left(\sum_{j=1}^{n} \|t_j\|_2^2 \right)^{1/2} \geqslant$$

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\pi} \sum_{j=1}^{n} \|t_j\|_1^2 \right)^{1/2} \geqslant \frac{\alpha}{2\pi} \sqrt{n},$$

$n = 1, 2, \ldots$. Therefore the order $\sqrt{n}$ on the right-hand side of the inequality (1) can not be decreased.

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On an inequality of Sidon

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For $f \in L^p(0, 2\pi)$ we set

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{[0,2\pi]} |f(x)| \qquad \text{for } p = \infty.$$

Let $\lambda > 1$ be a real number. We denote by $\Lambda(\lambda)$ the class of sequences $N = \{n_k\}_{k=1}^\infty$ of natural numbers such that $n_{k+1}/n_k \ge \lambda$, $k = 1, 2, \ldots$. If x is a real number, then $[x]$ denotes its integral part, and $\lceil x \rceil$ is the smallest integer n such that $n \ge x$. For a real number $r \ge 0$, let $\mathrm{T}(r)$ denote the space of all real trigonometric polynomials of the form

$$t(x) = A + \sum_{k=1}^{[r]} a_k \cos kx + b_k \sin kx$$

(by definition, the sum $\sum_{k=1}^0$ is put to be zero).

We prove the following result (see [1]) which generalizes a theorem due to Kashin and Temlyakov, which, in turn, generalizes the classical Sidon inequality.

Theorem. *Let $\varepsilon \in (0, 1)$ and $B \ge 1$ be real numbers, and let $N = \{n_k\}_{k=1}^\infty$ be a monotone increasing sequence of natural numbers such that N can be split into d sequences $N^{(j)} \in \Lambda(\lceil 7\sqrt{B} \rceil)$, $j = 1, \ldots, d$. Then for each trigonometric polynomial of the form*

$$f(x) = \sum_{k=l}^{m} p_k(x) \cos n_k x + q_k(x) \sin n_k x,$$

where $p_k, q_k \in \mathrm{T}(r_k),\ k = l, \dots, m,$

$$r_l = \min\left(\frac{n_{l+1} - n_l}{2(1+\varepsilon)}, \frac{n_l}{1+\varepsilon}\right),$$
$$r_k = \min\left(\frac{n_k - n_{k-1}}{2(1+\varepsilon)}, \frac{n_{k+1} - n_k}{2(1+\varepsilon)}, Bn_l\right), \quad k = l+1, \dots, m-1,$$
$$r_m = \min\left(\frac{n_m - n_{m-1}}{2(1+\varepsilon)}, Bn_l\right),$$

$m > l+1,\ l = 1, 2, \dots,$ *we have the inequality*

$$\|f\|_\infty \geq \frac{c}{d^2 \cdot \ln^2(1 + 1/\varepsilon)} \sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1,$$

where $c > 0$ *is an absolute constant.*

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On products of nuclear operators

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Let $0 < s \leqslant 1$. An operator $T : X \to Y$ in Banach spaces is called s-nuclear (nuclear if $s = 1$) if it admits a representation of the kind $Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$ for $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k||^s\, ||y_k||^s < \infty$. We use the notation $N_s(X, Y)$ for the linear space of such operators. We denote by S_p $(0 < p < \infty)$ the von Neumann-Schatten class of operators in Hilbert spaces.

We say that an operator T *can be factored through an operator* from S_p, if there exist a Hilbert space H and the operators $A \in L(X, H)$, $U \in S_p(H)$ and $B \in L(H, Y)$, such that $T = BUA$.

The following result was inspired by a question of B. S. Mityagin: Is it true that a product of two nuclear operators can be factored through an operator from S_1?

Theorem. *If $X_1, \ldots, X_{n+1}$ are Banach spaces, $s_k \in (0,1]$ and $T_k \in N_{s_k}(X_k, X_{k+1})$ for $k = 1, 2, \ldots, n$, then the product $T := T_n T_{n-1} \cdots T_1$ can be factored through an operator from S_r, where $1/r = 1/s_1 + 1/s_2 + \cdots + 1/s_n - (n+1)/2$.*

We will discuss the sharpness of this result. For example, we have:

1) If an operator T in a Banach space is nuclear and $m > 1$, then T^m can be factored through an operator from S_r, where $r = 2/(m-1)$.

2) There exists a nuclear operator T in the space $C[0,1]$ (or in the space $L_1[0,1]$) such that for any $m > 1$ and $r < 2/(m-1)$ the operator T^m can not be factored through an operator from S_r.

Luzin type properties, and the difference quotient set

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A function H is said to have the Luzin property N if the image of any null set under H is a null set. In this talk we will discuss the inverse property, of mapping positive mass sets to positive mass sets, especially when the function in question is $\mathrm{DQ}_f(x_1, x_2)$, the difference quotient of some function $f : \mathbb{R} \to \mathbb{R}$. We define the difference quotient set of f on a set E to be the set $\mathrm{DQ}_f(E) = \{\mathrm{DQ}_f(x_1, x_2) : x_1, x_2 \in E\}$, and show that under certain natural conditions, if E has positive mass, then $\mathrm{DQ}_f(E)$ will not only have positive mass, but contain an interval.

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c-Removable sets: Old and new results

Martin Rmoutil (speaker), Dušan Pokorný

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This work is still in progress. We study subsets of Euclidean spaces that are negligible from the point of view of convexity of functions (the "c" in c-removability comes from "convexity"). More precisely, a closed set $F \in \mathbb{R}^d$ is said to be *c-removable* if the following is satisfied: Whenever a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is locally convex on the complement of F, it is convex on the whole $\mathbb{R}^d$.

Five years ago, at the 37^{th} Summer Symposium in Real Analysis (although that particular one was technically a "Winter Symposium" as it took place in the Southern Hemisphere), I presented joint results with Dušan Pokorný disproving a conjecture by Jacek Tabor and Józef Tabor that c-removability is characterized by *interval thinness*, a notion that they introduced, which means that the set is essentially transparent in all directions: We found examples of sets which are c-removable, yet not intervally thin (one such example we call the *Holey Devil's Staircase*). We also found many examples of non-c-removable discontinua.

However, the question remained open of the existence of a nontrivial c-removable *continuum*. We now have such examples along with other new results, providing a better understanding of the notion of c-removability.

Sequences valued in convergence groups

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Convergence spaces are generalisation of topological spaces. We present the facts obtained while investigating the sequences valued in convergence groups with special reference to the boundedness of these sequences.

Some geometric properties in Banach spaces

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Some approximation theoretic characterizations of several geometric properties of Banach spaces will be presented. The characterizations are in terms of various types of approximative compactness. These characterizations provide some answers to a question posed by Deustch and Lambert in the paper "On continuity of metric projections", J. Approx. Theory, 1980.

Approximation by multivariate quasi-projection operators

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Quasi-projection operators

$$Q_j(f;\widetilde{\varphi},\varphi) = |\det M|^j \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}(M^j\cdot -k)\rangle \varphi(M^j\cdot -k),$$

where M is a matrix dilation, is studied for a class of band-limited functions φ and different classes of funcions/distributions $\widetilde{\varphi}$. An important special case is so-called Kantorovich-Kotelnikov type operators, where $\widetilde{\varphi}$ is a summable function. The L_p-rate of convergence for such operators is given in terms of the classical moduli of smoothness. Several examples of the Kantorovich-Kotelnikov operators generated by the multivariate sinc-function and the linear combinations of its translations are provided. Similar estimates in the weighted L_p spaces are obtained under additional assunption of some smootness of the Fourier transform of φ. This allows to estimate the error for reconstruction of signals (approximated functions) whose decay is not enough to be in L_p. Another special case is the sampling expansions, i.e., the case, where $\widetilde{\varphi}$ is the the Dirac delta-function. Approximation order in L_p-norm, $p \geqslant 2$, is investigated for such operators. These results are extended to the weighted L_p-norm for the Muckenhoup weights.

Some problems in harmonic analysis on compact zero-dimensional groups (non-abelian case)

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Here we extend some of our previous results obtained in [1] and [2] to the case of non-abelian groups.

Let G be a compact, 0-dimensional, metric group, *not necessarily abelian*, and let $\{G_n\}$ be a strictly decreasing sequence of open normal subgroups forming a neighborhood base at the identity. Let Σ denote the dual object of G, i. e., the set of equivalence classes of irreducible representations of G. If $\sigma \in \Sigma$, we pick a irreducible representation U^σ in the equivalence class σ. Let the representation U^σ act on the Hilbert space H^σ of the dimension d_σ. Note that all H^σ are of a finite dimension in our compact case. Annihilators of subgroups G_n in Σ are defined as $A_n = A(\Sigma, G_n) = (\sigma \in \Sigma : U^\sigma_x = I \text{ for all } x \in G_n)$.

For any additive complex measure μ on G and for any $\sigma \in \Sigma$ there exists a unique operator T_σ on H^σ such that $\langle T_\sigma \xi, \eta \rangle = \int_G \langle U^\sigma_{x^{-1}} \xi, \eta \rangle d\mu(x)$ for every $\xi, \eta \in H^\sigma$ (see [3]). Fourier-Stieltjes series of a measure μ is defined as

$$\sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(T_\sigma U^\sigma_x)$$

(here and below $\mathrm{tr}(\cdot)$ denotes the trace of an operator).

We say that a formal series

$$S \sim \sum_{\sigma \in \Sigma} d_\sigma \mathrm{tr}(B_\sigma U^\sigma_x), \tag{1}$$

where B_σ are bounded linear operators on H^σ, is convergent to a function f at $x \in G$ if its partial sums

$$\sum_{\sigma \in A_n} d_\sigma \mathrm{tr}(B_\sigma U^\sigma_x)$$

are convergent to $f(x)$ at x.

We prove that *if a series* (1) *is everywhere convergent to a finite function f then f is integrable on G in the sense of some generalization of Henstock integral and* (1) *is the Fourier-Stieltjes series of the measure $\mu = \int f$.*

Some extensions to the non-abelian case of results of [2] related to the properties of the sets of uniqueness are also obtained.

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On variational and Riemann approach to some Lusin-type integrals

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Soon after discovering his renowned integral (equivalent to the narrow Denjoy integral, or *Denjoy–Perron integral*) [2], Ralph Henstock, in a remark in his textbook [3], suggested a similar Riemann-type approach that ought to lead to the other of Denjoy's integral, now the wide Denjoy integral (or *Denjoy–Khintchine integral*). Since he provided there no detailed proof, this gave birth (much later) to a number of works on this problem [1, 4, 5]. The setting considered then was more general than Henstock's, with some generalized continuity (of primitives) used instead of ordinary continuity (present in the wide Denjoy approach).

Being inspired by a recent exposition by Brian Thomson on measure-theoretic characterizations of ACG and VBG properties [7], in the present work we use Thomson's weak and q-weak measures in characterizations of generalized wide Denjoy integrals, via absolute continuity of these measures. As a consequence, and a parallel result, we provide simplified Riemann definitions, equivalent to the definitions from [1, 4, 5].

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On the generalized binomial transform

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Attaching an additional sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ to the binomial transform, we obtain its extension that we use to define the generalized binomial transform $\mathcal{T}_\alpha$, which presents a correspondence between a set of infinitely continuously differentiable functions and a set of sequences linked to a generalized linear difference operator A_α. Taking different sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ gives rise to a family of transforms $\mathcal{T}_\alpha$.

We make use of $\mathcal{T}_\alpha$ to map derivatives to A_α, and integrals to A_α^{-1} as well. The inverse transform $\mathcal{B}_\alpha$ of $\mathcal{T}_\alpha$ is introduced and its properties are studied. Choosing the sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ such that $\alpha_n = (-1)^n$, it is shown that $\mathcal{B}_\alpha$ reduces to the Borel transform. Also, applying $\mathcal{T}_\alpha$ for the same sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ to Bessel's differential operator $D_x = \frac{d}{dx}x\frac{d}{dx}$, we obtain discrete Bessel's operator $\Delta n\nabla$.

As applications of Bessel's differential and discrete operator, it is shown that eigenfunctions of Bessel's operator are mapped to eigenvectors of discrete Bessel's operator, and models of population growth are considered.

On applications of the generalized binomial transform

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A sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ is associated with the generalized linear difference operator A_α, the generalized binomial transform $\mathcal{T}_\alpha$ and its inverse transform $\mathcal{B}_\alpha$, which present a correspondence between a set of infinitely continuously differentiable functions and a set of sequences. Taking different sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$, a family of transforms $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ is obtained.

We demonstrate the use of properties of the operator A_α, transforms $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ for mapping elementary functions, their derivatives and integrals to sequences. That enables us to map differential and integral equations and their solutions to difference equations and their solutions. We make use of the $\mathcal{T}_\alpha$-transform to map Bessel's differential operator $\frac{d}{dx}x\frac{d}{dx}$ to Bessel's difference operator $\Delta n\nabla$, and Bessel's differential equation and its solutions to a difference equation and its solutions. Apart from that, applications are concerned with obtaining solutions of differential equations of the classical orthogonal polynomials, and some combinatorial identities.

Convergence of Fourier series by Vilenkin system in the case of unlimited p(k)

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The paper generalizes an analogue of the Dini sign, proved in [1]. And also some consequences are deduced.

Let X denote the group of characters of zero-dimensional group G, which the second axiom of countability. Then X is a discrete, countable, abelian, torsion group. N.Y. Vilenkin [2] showed X is the union of subgroups $\{X_s\}_{s=0}^{\infty}, X_s \subset X_{s+1}$, such that X_{s+1}/X_s is of prime order p_s. Such a pair (G, X) is called a Vilenkin system.

An analogue of the Dini sign with $p_k < +\infty$ has been proved [1]. In this work we prove an analogue of the Dini sign for unbounded p_k. And we prove convergence of Fourier series by Vilenkin system for $f(g) \in Lip\ \alpha(G)$ for unbounded p_k as well.

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