“IRRATIONAL” PROJECTIONS AND MUCKENHOUPT CONDITIONS

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In analysis of Systems of dilated functions [1], we faced a few questions on “irrational” projections in weighted L^p–spaces.

The Hilbert – Riesz projection

$$
Q : e^{ikx} \to \lambda(k) e^{ikx}, \quad k \in \mathbb{Z}
$$

$$
\lambda(k) = 1 \text{ if } k \geq 0, \qquad (1)
$$

$$
= 0 \text{ if } k < 0,
$$

is bounded in $L^p(\mathbb{T})$, $\mathbb{T} = [0, 2\pi)$, $1 < p < \infty$. If we consider weighted L^p spaces $L^p(\mathbb{T}; w)$, w, $\frac{1}{w} \in L^1(\mathbb{T})$, $w \geq 0$, with the norm

$$
\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p w(x) \frac{dx}{2\pi} \right)^{1/p},
$$

the Muckenhoupt condition (A_p) [2] on the weight w is necessary and sufficient for boundedness of the projection Q in $L^p(\mathbb{T}; w)$, $1 < p < \infty$.

In the multidimensional case, as an analogue of a projection Q, for any non-zero linear function ℓ on $\mathbb{R}^d$, $d \geq 2$,

$$
\ell(\xi) = \sum_{j=1}^{d} \lambda_j \xi_j, \quad (\lambda_j) \in \mathbb{R}^d \setminus \{0\}, \qquad (2.0)
$$

we define the projection Q_λ,

$$
Q_\lambda : \begin{array}{ll} e^{i\langle k,x \rangle} \to e^{i\langle k,x \rangle} & \text{if } \ell(k) \geq 0, \\ \to 0 & \text{if } \ell(k) < 0, \end{array} \quad k \in \mathbb{Z}^d, x \in \mathbb{T}^d. \qquad (2.1)
$$

If all coefficients λ_j, $1 \leq j \leq d$, are rational, the question about boundedness of Q_λ on $L^p(\mathbb{T}^d; w)$ can be reduced to the case $(\lambda_j)_{j=1}^d = (e_1)$, i.e., $\lambda_1 = 1$, $\lambda_j = 0$, $2 \leq j \leq d$, which is essentially a one-dimensional case, and the complete answer is given with proper adjustment of the conditions (A_p), $d = 1$.

Now I want to ask a couple of questions on “irrational” half-spaces, or projections Q_λ, when the vector $\lambda = (\lambda_j)_{j=1}^d$ is *irrational*, i.e., its coordinates are linearly independent over the rationals.. It seems the difficulties come because the boundedness of this “discrete” operator cannot be made equivalent to the boundedness

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of some "continuous" singular operator as it was in the case (1), or even for $d \geq 2$ if ℓ is rational. Usually, we use the operator

$$(Hf)(e^{i\vartheta}) = \frac{1}{2\pi}\,\mathrm{p.\,v.}\int_{-\pi}^{\pi} f(e^{it}) \cot\frac{\vartheta - t}{2}\,dt, \qquad \text{or} \qquad g \mapsto \mathrm{p.\,v.}\int\limits_{\mathbb{R}} \frac{g(\xi)}{\xi - x}\,dx.$$

Question 1. Let ℓ or λ, in (2.0) and (2.1) be *irrational*. Give necessary and sufficient conditions [on a weight $w \geq 0$, $w, \frac{1}{w} \in L^1(\mathbb{T}^d)$] of the boundedness of the projections Q_λ in $L^p(\mathbb{T}^d; w)$.

In the case $w \equiv 1$, i.e., for Lebesgue measure m, we can show that Q_λ is bounded in $L^p(\mathbb{T}^d; m)$, $1 < p < \infty$, for any ℓ in (2.0); moreover, these projections are uniformly bounded, i.e., $\|Q_\lambda | L^p(\mathbb{T}^d; m)\| \leq C(p) < \infty$, with $C(p)$ independent of λ.

The case of weights

$$w(x) = \begin{cases} \sum_{j=1}^{d} |x_j|^{a_j}, & \text{if } \sum_{j=1}^{d} |x_j|^2 \leq \frac{1}{9} \\ 1, & \text{otherwise} \end{cases} \tag{3}$$

could be of special interest.

Question 2. Let ℓ, or λ, be *irrational*, and $w(x) \in (3)$ with $a_j > 0$, $1 \leq j \leq d$, and $\sum_{j=1}^{d} \frac{1}{a_j} > 1$. For which set $(a_j)_{j=1}^d$ of potentials is the projection Q_λ bounded in $L^p(\mathbb{T}^d, w)$, $1 < p < \infty$?

Question 3. Let us restrict ourselves to the L^2–case only, and try to answer the above questions (Que. 1 and 2) for $p = 2$.

Example 4. If $p_1 = 2, p_2 = 3, \ldots, p_n$ is the sequence of primes, then for any d the set

$$\lambda_j^* = \log p_j, \quad 1 \leq j \leq d, \quad d \geq 4,$$

is *irrational*, that is, linearly independent over the rationals. Let

$$w^*(x) = \begin{cases} x_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{d} x_j^4, & \text{if } \sum_{j=1}^{d} |x_j|^2 \leq \frac{1}{9}, \\ 1, & \text{otherwise.} \end{cases}$$

Show that the projection Q_{λ^*} is *not* bounded in $L^2(\mathbb{T}^d; w^*)$.

[1] B. Mityagin, "Systems of dilated functions: Completeness, minimality, basisness," *Func. Anal. Appl.* 51:3 (2017), p. 236 – 239.

[2] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Trans. Amer. Math. Soc.* 165 (1972), p. 207 – 226.